

WKB — Not So Bad After All

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It was found recently that tunneling probabilities over a barrier is roughly twice as large as that given by standard WKB formula. Here we explained how this come from and showed that WKB method does give a good approximation over almost entire energy range provided that we use appropriate connection relations.

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I. Introduction

WKB method was first invented by Jeffreys¹ and was applied to solve Schrödinger equation by Wentzel, Kramers and Brillouin². It is a powerful tool and has many applications, e. g. , waves in an inhomogeneous plasma³. Consider a second order ordinary differential equation:

$$\psi''(x) + k^2(x)\psi(x) = 0, \tag{1}$$

where prime denotes differentiation with respect to argument. The function k^2 may be negative for some x . In this case, we define $\kappa^2(x) \equiv -k^2(x)$. WKB method approximates solutions of (1) by combinations of

$$\exp\left(\pm i \int^x k(y) dy\right) / \sqrt{k(x)}, \tag{2}$$

provided that

$$|k'/k^2| \ll 1 \tag{3}$$

is satisfied. If (3) is not satisfied over a range (or at a point), then we need to connect WKB solutions of both sides appropriately by some connection relations. One common situation where (3) breaks down is that $k^2 \approx 0$ near a turning point.

In a recent paper⁴, a quantum mechanical tunneling example is considered. The Schrödinger equation in this one dimensional problem can be

represented by (1) with

$$k^2(x) = \begin{cases} E - \alpha x^2 & \text{for } 0 \leq x \leq b \\ E & \text{otherwise,} \end{cases} \quad (4)$$

where $b = \sqrt{E_b/\alpha}$ and E , E_b and α are constants. E may be assigned the physical meaning of energy. It was shown that, for $E < E_b$, the true (numerical) tunneling probabilities T_n (see Appendix for numerical method) are roughly twice as large as the well known WKB tunneling factor:

$$T_w = e^{-2 \int_a^b \kappa(x) dx} = e^{\frac{E}{\sqrt{\alpha}} \left\{ \ln[\beta + \sqrt{\beta^2 - 1}] - \beta \sqrt{\beta^2 - 1} \right\}}, \quad (5)$$

where $a = \sqrt{E/\alpha}$ and $\beta = \sqrt{E_b/E}$. Values of T_w and T_n for different E are shown in Table I and Table II.

This result was unexpected to us at first. After a little survey over some standard textbooks, it was found that two books did warn about such an error^{5,6}, although without elaborations. Another book did an example with correct connection relation⁷, but did not give general discussions. Two other books used the standard tunneling factor without warning^{8,9}, although one of them indicated that it is just a rough approximation⁹. In this case, we have reasons to assume that the restrictions in applying the standard WKB formula are not very well known to students. Therefore, we would like to show how to apply WKB method to solve a tunneling problem more carefully.

In section II, we briefly review the derivation of the standard WKB tunneling factor using standard connection relations, which are also useful for

our discussions later, over a typical turning point. In section III, the tunneling factor is corrected when one turning point is a sharp edge. In section IV, further correction is made when the tunneling probability is not small. Section V considers the case that $E \rightarrow 0$. Section VI deals with the case that $E \approx E_b$. Finally section VII treats the case that $E > E_b$. It will be shown that WKB method does give good approximation to the tunneling probability if we use different connection relations for different energy range appropriately.

II. Standard WKB tunneling formula

Consider (1) with x near a turning point $x = 0$. Assume that $k^2(x)$ can be approximated by:

$$k^2(x) \approx \left. \frac{dk^2}{dx} \right|_{x=0} x. \quad (6)$$

Let $(dk^2/dx)_0 > 0$ for the time being. Solutions of (1) with k^2 given by (6) can be written as combinations of Airy's functions¹⁰ $\text{Ai}(-\lambda x)$ and $\text{Bi}(-\lambda x)$ with $\lambda \equiv |(dk^2/dx)_0|^{1/3}$. Then the asymptotic behaviors of Ai and Bi give the standard WKB connection relations¹⁰:

$$\begin{array}{ccccc} -\infty & \longleftarrow & x & \longrightarrow & \infty \\ \frac{1}{2\sqrt{\kappa}} e^{-\int_x^0 \kappa dy} & \longleftarrow & \sqrt{\frac{\pi}{\lambda}} \text{Ai}(-\lambda x) & \longrightarrow & \frac{1}{\sqrt{k}} \sin(\int_0^x k dy + \frac{\pi}{4}) \\ \frac{1}{\sqrt{\kappa}} e^{\int_x^0 \kappa dy} & \longleftarrow & \sqrt{\frac{\pi}{\lambda}} \text{Bi}(-\lambda x) & \longrightarrow & \frac{1}{\sqrt{k}} \cos(\int_0^x k dy + \frac{\pi}{4}). \end{array} \quad (7)$$

For $(dk^2/dx)_0 < 0$, we only need to change λ to $-\lambda$ in (7), interchange the limits of integrations and interchange left and right hand sides. These standard connection relations are valid only if the turning point is smooth enough, i.e., we require (6) to be valid up to some x where (3) is also satisfied. This is not always possible³.

Consider a barrier with $k^2 > 0$ for $x \leq a$ and $x \geq b$, $k^2 < 0$ for $a \leq x \leq b$ and that WKB condition (3) is valid over the whole range of x except near the two smooth turning points a and b . Then for a incident wave from $x = -\infty$ direction, we have only outgoing wave for $x > b$:

$$\psi = \frac{1}{\sqrt{k}} e^{i \int_b^x k dy}. \quad (8)$$

By (7), ψ connects to:

$$\psi = \left[\frac{1}{\sqrt{\kappa}} e^{\int_x^b \kappa dy} + \frac{i}{2\sqrt{\kappa}} e^{-\int_x^b \kappa dy} \right] e^{-i\pi/4} \quad (9)$$

for $a < x < b$. Then for x close to a ,

$$\psi \approx \frac{e^I}{\sqrt{\kappa}} e^{-\int_a^x \kappa dy - i\pi/4}, \quad (10)$$

where $I \equiv \int_a^b \kappa dy$ and is assumed large. By (7) again, ψ connects to

$$\psi = 2 \frac{e^{I-i\pi/4}}{\sqrt{k}} \sin \left(\int_x^a k dy + \pi/4 \right) \quad (11)$$

for $x \leq a$. Since the sine term is just a combination of incoming and reflected wave with equal intensity, we have reflection coefficient $R_w \approx 1$ and tunneling probability $T_w = e^{-2I}$. This is how (5) come from.

From the above derivation, we see that in order to apply the standard WKB tunneling factor (5), three conditions must be satisfied :

- (i) there are two and only two turning points;
- (ii) (3) is valid except near the two turning points where k^2 can be approximated by (6); and
- (iii) $T_w \ll 1$.

We immediately see that the potential given by (4) does not satisfy (ii) since k^2 is discontinuous at $x = b$. This is the main reason for the factor of two error. Also, for larger E , T_w is not very small, (iii) is also violated.

III. Sharp turning point correction

Let us now consider k^2 with a discontinuity at $x = b$, so that $k^2 = -\kappa_b^2 < 0$ at $b-0^+$ and $k^2 = k_b^2 > 0$ at $b+0^+$. We still assume that the WKB condition (3) remains valid except at b and near the smooth turning point $x = a$. For $x > b$, there is only outgoing wave (8). It connects to

$$\psi = \frac{1}{2\sqrt{\kappa}} \left\{ \left[\sqrt{\frac{\kappa_b}{k_b}} - i\sqrt{\frac{k_b}{\kappa_b}} \right] e^{\int_x^b \kappa dy} + \left[\sqrt{\frac{\kappa_b}{k_b}} + i\sqrt{\frac{k_b}{\kappa_b}} \right] e^{-\int_x^b \kappa dy} \right\} \quad (12)$$

for $a < x \leq b$. We can check whether (12) is correct by substituting $x = b$ into (8) and (12). We should see that ψ and ψ' are indeed continuous. Similar to steps (9) to (11) and still assuming I to be large, the corrected tunneling

probability can be found as:

$$T_1 = \frac{4\kappa_b k_b}{\kappa_b^2 + k_b^2} T_w. \quad (13)$$

For k^2 given by (4),

$$T_1 = \frac{4\sqrt{E_b/E - 1}}{E_b/E} T_w. \quad (14)$$

It can be easily shown by (13) that $T_1 \leq 2T_w$. The equal sign holds for $k_b = \kappa_b$. This explains the factor of two found by ref. 4, Some values of T_w and T_1 are shown in Table I to compare with true (numerical) tunneling probabilities T_n . Parameters $\alpha = 0.040965$, $E_b = 1.2776$ were chosen so that we may compare with Table I of ref. 4. We see that T_1 is much closer to T_n from $E \approx 0.2$ to $E \approx 0.6$ while T_w is nearly a factor of two smaller. However, for large E when T_w larger than 0.2, T_1 also fails since we assumed T_w to be small in the above derivation. This error will be corrected in next section.

IV. Finite T_w correction

First, let us consider the correction of T_w itself. If we keep both terms in (9), then ψ will connects to

$$\psi = \frac{1}{\sqrt{kT_w}} \left[2\sin(\theta) + \frac{i}{2}T_w \cos(\theta) \right] e^{-i\pi/4} \quad (15)$$

for $x < a$ where $\theta \equiv \int_x^a k dy + \pi/4$. Then by finding the coefficient of incoming term, the corrected tunneling probability can be found as:

$$T_{w2} = T_w / (1 + T_w/4)^2. \quad (16)$$

Similarly, keeping both terms in (12) gives correction of T_1 :

$$T_2 = T_1 / \left(1 + T_1/2 + T_w^2/4 \right). \quad (17)$$

Some values of T_2 are also shown in Table I and Table II. We see that $T_2 \rightarrow T_1$ as $T_w \rightarrow 0$. But for large T_w , T_2 differs from T_1 quite a lot and gives better approximation to T_n up to $E \approx 1$, or $T_w \approx 0.65$, which is quite large. However, for $E \rightarrow E_b$, T_2 also fails. For example, for $E = E_b$, both T_1 and T_2 equal to zero while T_n is actually quite large. The reason for this error is that we assumed that WKB condition (3) is satisfied up to $x = b$. This is not true for $E \rightarrow E_b$, since κ is small, even for $x = b$, while κ' is not small. So we cannot use (13) or (17) for E close to E_b . Another way to apply WKB method for this case is discussed in section VI.

In the other extreme, the fact that for $E \rightarrow 0$, $T_w \rightarrow$ finite value while $T_n \rightarrow 0$, is another limitation of the standard WKB tunneling factor⁴. Although T_1 and T_2 give the right value, i.e. zero, at $E = 0$ as shown in Table I, the dependence of E is wrong as we can see from Table II. We see that $T_n \propto E$ as $E \rightarrow 0$ while $T_2 \approx T_1 \propto \sqrt{E}$. So, there are extremely large error between T_n with T_1 or T_2 . We will discuss this case in the next section.

V. For $E \rightarrow 0$

Consider the general case first. Let $k^2 = E \rightarrow 0$ for $x \leq a$. By (12) and connection formula (7), we know

$$\psi \approx \frac{1}{\sqrt{T_w}} \left[\sqrt{\frac{\kappa_b}{k_b}} - i \sqrt{\frac{k_b}{\kappa_b}} \right] \sqrt{\frac{\pi}{\lambda}} \text{Ai}(\lambda x) \quad (18)$$

for $x \rightarrow a^+$, where $\lambda^3 \equiv -(dk^2/dx)_{x=a}$. Again, T_w small was assumed, although this assumption can be removed if higher accuracy is desired. Now assume

$$\psi = C e^{i\sqrt{E}x} + D e^{-i\sqrt{E}x} \quad (19)$$

for $x < a \rightarrow 0$, where C and D are constants. Matching (18) and (19) at $x = 0$ and using the fact that $E \rightarrow 0$, we got the tunneling probability:

$$T = \frac{1}{|C|^2 \sqrt{E}} = \frac{\sqrt{E}}{\pi \lambda [\text{Ai}'(0)]^2} T_1. \quad (20)$$

Note that power series expansions for Airy's functions are¹⁰:

$$\begin{aligned} \text{Ai}(z) &= c_1 f(z) - c_2 g(z), \\ \text{Bi}(z) &= \sqrt{3} [c_1 f(z) + c_2 g(z)], \end{aligned} \quad (21)$$

where

$$\begin{aligned} f(z) &= 1 + \frac{z^3}{2 \cdot 3} \left\{ 1 + \frac{z^3}{5 \cdot 6} \left[1 + \frac{z^3}{8 \cdot 9} (1 + \cdots) \right] \right\}, \\ g(z) &= z \cdot \left(1 + \frac{z^3}{3 \cdot 4} \left\{ 1 + \frac{z^3}{6 \cdot 7} \left[1 + \frac{z^3}{9 \cdot 10} (1 + \cdots) \right] \right\} \right), \\ c_1 &= 0.355028, \\ c_2 &= 0.258819. \end{aligned}$$

So, $\text{Ai}'(0) = -c_2$. In the derivation of (20), we assumed λ finite and k^2 smooth enough near $x = a$. However, for k^2 given by (4), $\lambda = (4\alpha E)^{1/6} \rightarrow 0$. This means that $T \propto E^{5/6}$. This is not correct since $T_n \propto E$. The reason is that as $E \rightarrow 0$, $(dk^2/dx)_a \rightarrow 0$ while $(d^2k^2/dx^2)_a = -2\alpha$ is constant. This means that (6) is not valid and we cannot approximate solutions by Airy's functions. In order to correct this, we need to consider specifically the potential given by (4). Now, as $E \rightarrow 0$, $k^2 \rightarrow -\alpha x^2$ for $0 \leq x \leq b$. Use a change of variable $y = \alpha^{1/4}x$, (1) becomes:

$$\psi_{yy} - y^2\psi = 0. \quad (22)$$

This equation can be solved by so called parabolic cylinder functions¹⁰. Actually, we may use these functions to solve (1) with k^2 given by (4) exactly and write the tunneling probability in closed form. This is out of the scope of this paper. So let us assume that we do not know these functions. We will see that we do not need to solve (22) exactly. Instead, we may study it by WKB method! First, note that $(dy/dy)/y^2 = 1/y^2 \ll 1$ as $y \rightarrow \infty$, i.e. the WKB condition (3) is satisfied asymptotically. So (22) has asymptotic solutions given by (2) with $k^2 = -y^2$. Let us define two solutions of (22) by their asymptotic behaviors as $y \rightarrow \infty$:

$$\begin{aligned} \text{Aj}(y) &\rightarrow \frac{1}{\sqrt{y}} e^{-\int_0^y z^2 dz} = \frac{1}{\sqrt{y}} e^{-y^2/2}, \\ \text{Bj}(y) &\rightarrow \frac{1}{\sqrt{y}} e^{\int_0^y z^2 dz} = \frac{1}{\sqrt{y}} e^{y^2/2}. \end{aligned} \quad (23)$$

Then we may write the solution for $0 \leq x \leq b$ as

$$\psi = [\text{Bj}'(\eta)\text{Aj}(y) - \text{Aj}'(\eta)\text{Bj}(y)]/2, \quad (24)$$

where $\eta \equiv \alpha^{1/4}b$, in order to match with an outgoing solution $e^{i\sqrt{E}x} \rightarrow 1$ for $x \geq b$. Note that we have used the fact that the Wronskian $W = \text{AjBj}' - \text{Aj}'\text{Bj} = 2$ and that $E \rightarrow 0$. Using similar steps as (19) to (20) and neglecting $\text{Aj}'(\eta)$ as compared with $\text{Bj}'(\eta)$, the tunneling probability can be found as:

$$T_0 = 16E / \left\{ \alpha^{1/2} [\text{Aj}'(0)\text{Bj}'(\eta)]^2 \right\}. \quad (25)$$

This gives the correct E dependence since α and b are independent of E . To find $\text{Bj}'(\eta)$, we may use (23) as a first approximation:

$$\text{Bj}'(y) \approx \sqrt{y}e^{y^2/2}. \quad (26)$$

So,

$$T_0 = \frac{16E}{\alpha^{1/4}E_b^{1/2} [\text{Aj}'(0)]^2} T_w. \quad (27)$$

This gives the dependence on E , α and E_b since $\text{Aj}'(0)$ is only a constant. In order to compare T_0 with T_n numerically, we need to know $\text{Aj}'(0)$, which can be found by numerical integration of (22). I found¹¹ $\text{Aj}'(0) \approx -0.9777$. Using this, some values of T_0 are shown in Table II. We can see that they are quite close, although there are more than 10% difference. The main error can be shown to be due to the approximation in (26). To see this, we note that the factor before the exponential function in (23) should actually be an

asymptotic series¹⁰. The series of Bj can be found by requiring cancellations between terms when it is put into (22):

$$\text{Bj}(y) \longrightarrow \frac{e^{y^2/2}}{\sqrt{y}} \left(1 + \frac{3}{4 \cdot 4y^2} \left\{ 1 + \frac{5 \cdot 7}{8 \cdot 4y^2} \left[1 + \frac{9 \cdot 11}{12 \cdot 4y^2} (1 + \dots) \right] \right\} \right). \quad (28)$$

This series can be evaluated numerically up to a term with smallest magnitude. This brought a factor of 1.133 to T_0 in Table II, e.g. T_0 changed from 5.97×10^{-10} to 6.76×10^{-10} for $E = 10^{-8}$. This is very close to T_n which is 6.78×10^{-10} .

The dependence of α and E_b in (27) were also be verified numerically by choosing different α and E_b . In general, as α decreases or E_b increases, i.e. $T_w \rightarrow 0$, the difference between T_0 and T_n decreases. The fact that we can find out the dependence of the tunneling probability on E , α and E_b by using WKB idea without solving (22) shows how powerful WKB method may be if applied correctly.

VI. For $E \approx E_b$

For $E \approx E_b$, (14) and (17) no longer give values close to T_n as we can see from Table I. However, we still can calculate the tunneling probability by WKB method. Now, $b - a$ is small, we may approximate (4) by $k^2 \approx -\lambda^3(x - a)$ for x in a region near a , including b , where $\lambda^3 = 2(\alpha E)^{1/2}$. Then for this region,

the solution is given by:

$$\psi = \pi e^{i\sqrt{E}b} \left\{ \left[\text{Bi}'(\xi) - \frac{i}{\lambda} \sqrt{E} \text{Bi}(\xi) \right] \text{Ai}(z) + \left[-\text{Ai}'(\xi) + \frac{i}{\lambda} \sqrt{E} \text{Ai}(\xi) \right] \text{Bi}(z) \right\}, \quad (29)$$

where $z \equiv \lambda(x-a)$, $\xi \equiv \lambda(b-a)$, in order to match with an outgoing solution $e^{i\sqrt{E}x}$ for $x \geq b$. Using standard connection formula (7), we may connect it to WKB type solutions (2) for $x \rightarrow -\infty$. Then, by grouping the coefficient of the incoming terms, we found the tunneling probability:

$$T_b = \frac{4\lambda\sqrt{E}/\pi}{\left[\text{Bi}(\xi)\sqrt{E} - \lambda\text{Ai}'(\xi) \right]^2 + \left[\text{Ai}(\xi)\sqrt{E} + \lambda\text{Bi}'(\xi) \right]^2}. \quad (30)$$

The Airy's functions can be evaluated by power series expansions (21). Some values of T_b are shown in Table I for $E \leq E_b$ and in Table III for $E > E_b$. We see that T_b gives a quite good approximation to T_n for $|E - E_b| < 0.7$. The range of validity for this approximation is surprisingly large at first sight. However, if we remember that the potential is proportional to x^2 so that although $|E - E_b|$ is not small, $|a - b|$ may be small enough for the approximation to work.

VII. For $E > E_b$

For the case that E larger than E_b , $T_w = 1$ while the true “tunneling probability” (it may be better to call it transmission coefficient now) may differ from 1 quite a lot. However, WKB method still gives a good approximation.

We now assume that WKB condition (3) is satisfied for $x \leq b$ all the way to $-\infty$. Then, we only need to match the outgoing solution $e^{i\sqrt{E}x}$ for $x \geq b$ with WKB type solutions (2) for $x \leq b$. Find the coefficient of the incoming term. Then the transmission coefficient can be found

$$T_\infty = \frac{4\sqrt{E(E - E_b)}}{[\sqrt{E - E_b} + \sqrt{E}]^2}. \quad (31)$$

Some values of T_∞ are shown in Table III. Since both T_∞ and $T_n \rightarrow 1$ as $E \rightarrow \infty$. It is more appropriate to compare reflection coefficients $R_\infty \equiv 1 - T_\infty$ and $R_n \equiv 1 - T_n$. We see that R_∞ gives a very good approximation to R_n for large E . For E close to E_b , T_∞ fails and we need to use T_b instead.

VIII. Conclusions

From the above discussions, we see that the standard WKB tunneling factor T_w fails badly for $E \rightarrow 0$, $E \approx E_b$ and may have error up to factor of two in between. However, WKB approximation not necessarily fail provided that we use appropriate connection relations. For the example that k^2 given by (4), WKB approximation works for almost entire energy range from $E = 0$ to $E \rightarrow \infty$ if we use different connection relations for different ranges of energy. Our conclusion is that when we use the standard WKB formula T_w , we need to be very careful. If higher accuracy is desired, we need to consider connection relations case by case.

Appendix: Numerical calculation of T_n

Although a numerical method to find T_n was described in ref. 4, a different method was used in this paper. Let

$$\begin{aligned}\psi(b) &= e^{i\sqrt{E}b} \\ \psi_x(b) &= i\sqrt{E}e^{i\sqrt{E}b},\end{aligned}$$

so that ψ connects to an outgoing solution $e^{i\sqrt{E}x}$ for $x \geq b$. Use Runge–Kutta method to integrate (1), with k^2 given by (4), from $x = b$ back to $x = 0$. Match with incoming and reflected waves for $x \leq 0$ and find the coefficient of the incoming term. Then the transmission coefficient can be found by

$$T_n = 4 \left/ \left| \psi(0) - \frac{i}{\sqrt{E}} \psi_x(0) \right|^2 \right.$$

Table I.

Comparison of tunneling probabilities, T_w, T_1, T_2 and T_b calculated by WKB method using different approximations with T_n calculated numerically, for energy E in the middle range. Parameters $\alpha = 0.040965, E_b = 1.2776$ were chosen so that we may compare with Table I of ref. 4.

E	T_w	T_1	T_2	T_b	T_n
0	0.00181	0	0	—	0
0.1	0.00611	0.00656	0.00654	0.0342	0.00987
0.2	0.0144	0.0210	0.0208	0.0543	0.0285
0.3	0.0297	0.0504	0.0491	0.0896	0.0609
0.4	0.0555	0.103	0.0979	0.142	0.112
0.5	0.0962	0.188	0.171	0.214	0.187
0.6	0.156	0.312	0.269	0.301	0.283
0.7	0.240	0.479	0.382	0.397	0.391
0.8	0.351	0.678	0.495	0.494	0.500
0.9	0.486	0.886	0.590	0.584	0.597
1.0	0.640	1.06	0.647	0.663	0.678
1.1	0.799	1.11	0.646	0.729	0.741
1.2	0.938	0.896	0.537	0.783	0.791
E_b	1	0	0	0.817	0.821

Table II.

Comparison of tunneling probabilities, T_w , T_2 and T_0 , calculated by WKB method using different approximations with T_n calculated numerically, for energy $E \rightarrow 0$ ($\alpha = 0.040965$, $E_b = 1.2776$).

E	T_w	T_2	T_0	T_n
10^{-8}	1.81×10^{-3}	6.42×10^{-7}	5.97×10^{-10}	6.78×10^{-10}
10^{-7}	1.81×10^{-3}	2.03×10^{-6}	5.97×10^{-9}	6.78×10^{-9}
10^{-6}	1.81×10^{-3}	6.42×10^{-6}	5.97×10^{-8}	6.78×10^{-8}
10^{-5}	1.81×10^{-3}	2.03×10^{-5}	5.97×10^{-7}	6.78×10^{-7}
10^{-4}	1.82×10^{-3}	6.44×10^{-5}	5.99×10^{-6}	6.78×10^{-6}
10^{-3}	1.86×10^{-3}	2.08×10^{-4}	6.11×10^{-5}	6.80×10^{-5}
10^{-2}	2.17×10^{-3}	7.64×10^{-4}	7.11×10^{-4}	7.04×10^{-4}
10^{-1}	6.11×10^{-3}	6.54×10^{-3}	1.93×10^{-2}	9.87×10^{-3}

Table III.

Comparison of tunneling probabilities, T_b and T_∞ , calculated by WKB method using different approximations with T_n calculated numerically, for energy $E > E_b$. $R_\infty = 1 - T_\infty$ and $R_n = 1 - T_n$ are reflection coefficients ($\alpha = 0.040965$, $E_b = 1.2776$).

E	T_b	T_∞	T_n	R_∞	R_n
1.3	0.826	0.410	0.829	0.509	0.171
1.6	0.907	0.855	0.901	0.145	0.0986
2.0	0.956	0.938	0.947	0.0621	0.0528
2.6	0.985	0.972	0.974	0.0280	0.0262
3.0	1.01	0.981	0.982	0.0190	0.0184
4.0	—	0.991	0.991	0.00920	0.00901
5.0	—	0.995	0.995	0.00542	0.00539
6.0	—	0.996	0.996	0.00357	0.00356
7.0	—	0.997	0.997	0.00253	0.00252

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- ¹⁰ M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington D. C., 1964).
- ¹¹ With the knowledge of parabolic cylinder functions, it can be shown that $A_j'(0) = -2\pi^{1/2}/\Gamma(1/4) \approx -0.97774$.